

# A Pinching constant for Harmonic Manifolds

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## Abstract

In this note we shall show that the sectional curvature of a harmonic manifold is bounded on both sides. In fact we shall give a pinching constant for all harmonic manifolds. We shall use the imbedding theorem for harmonic manifolds proved by Z.I.Szabo and the description of screw lines in hilbert spaces to prove the result.

## 1 Introduction

Let  $(M, g)$  be a riemannian manifold. It is well known that radial harmonic functions, i.e solutions to  $\Delta f = 0$  which depends only on the geodesic distance  $r(x, .)$ , exists on  $M$  only when the density funtion  $\omega_p = \sqrt{|\det(g_{ij})|}$  in a normal coordinate neibourhood around each point  $p$  depends only on the geodesic distance  $r(p, .)$ . A riemannian manifold is said to be harmonic if the density funtion satisfies the above radial property. The only known examples of Harmonic spaces (for quite some time ) were the rank one symmetric spaces. The Lichnerowicz conjecture asserted that these are the only ones. In [4] Szabo proves that any compact harmonic manifold with finite fundamental group is rank one symmetric. Later Damek and Ricci [2] gave examples of noncompact harmonic manifolds which are not rank one symmetric. Apart from these there are no known examples of harmonic spaces. The sectional curvature of a harmonic space was known to be bounded [1], our result gives the pinching constant.

## 2 The main Result

Let  $(M, g)$  be a riemannian manifold. Let  $(x_1, \dots, x_n)$  be a normal co-ordinate neighbourhood around a point  $p \in M$ . The function

$$\omega_p = \sqrt{|\det(g_{ij})|}$$

is the volume density of  $(M, g)$ . The density function in polar co-ordinates  $(r_p, \phi)$  is then given by  $\theta_p = r_p^{n-1} \omega_p$ , where  $r_p$  is the geodesic distance from  $p$  and  $\phi$  is a point on the unit sphere. A riemannian manifold is said to be harmonic if the density function  $\theta_p$  is a function of the geodesic distance  $r(p, \cdot)$  alone. Besse [1] constructed isometric imbeddings of compact harmonic manifolds into their eigen spaces. In [4] Szabo generalised Besse's imbedding theorem. We shall state the generalised version. Consider a  $C^1$  function  $h : R_+ \rightarrow R$  with  $h'(0) = 0$  such that  $h, h' \in L^2_\theta(R)$ . We define the map

$$\Phi_h : M \rightarrow L^2(M) \text{ by}$$

$$\Phi_h(p) = h_p, \text{ where } h_p(y) = h(r(p, y)).$$

**Theorem 1** *Let  $(M, g)$  be a harmonic manifold. Then*

1. *For any function  $h$  as above the map*

$$\Phi : M \rightarrow L^2(M); \quad p \mapsto h_p$$

*where  $h_p(y) = h(r(p, y))$  is an isometric immersion of the harmonic space  $M$  into a sphere of  $L^2(M)$ .*

2. *The geodesics of  $\Phi(M)$  are congruent screw lines in the space  $L^2(M)$ . By screw lines in  $L^2(M)$  we mean a rectifiable continuous curve  $r(s)$  parametrized by arclength  $s$  for which the distance  $d(r(t), r(s))$  in  $L^2(M)$  depends only on the arclength  $t - s$  for any two points  $r(t), r(s)$ .  $\square$*

we shall first study the second fundamental form of a harmonic manifold. Neumann and Shoenberg [3] constructed for any screw line  $r(t)$  in a hilbert space  $H$  a continuous one parameter family of unitary operators  $U(t) = e^{tX}$ ,  $X$  a skew symmetric operator such that  $r(t) = e^{tX}v, v = r(0)$ . hence the general equation of a screw line is:

$$\gamma(t) = e^{tX}v \text{ for a skew symmetric operator } X.$$

Let  $B$  be the second fundamental form of  $M$ . As  $\gamma$  is a geodesic,

$$\begin{aligned} B(\gamma', \gamma') &= \frac{d^2\gamma}{dt^2} = X^2 e^{tX} v \\ \text{hence } |B(\gamma', \gamma')|^2 &= |X^2 e^{tX} v|^2, \text{ which is a constant.} \end{aligned}$$

If  $\sigma$  is another geodesic congruent to  $\gamma$ , then  $\gamma = U\sigma$ , where  $U$  is a unitary operator. Hence  $B(\gamma', \gamma') = UB(\sigma', \sigma')$ , hence  $|B(\gamma', \gamma')|^2 = |B(\sigma', \sigma')|^2$ . The next result is a general result on bilinear forms.

**Lemma 2.1** *Let  $B : V \times V \rightarrow W$  be a symmetric bilinear form, where  $V, W$  are inner-product spaces over  $\mathbb{R}$ , such that  $\|B(u, u)\| = c\|u\|^2$ , where  $c$  is a constant, then*

$$2\|B(u, v)\|^2 = c^2(\|u\|^2 + \|v\|^2 + 2(u, v)^2) - (B(u, u), B(v, v))$$

.

**Proof:**  $4B(u, tv) = B(u + tv, u + tv) - B(u - tv, u - tv)$ , taking norms one gets

$$\begin{aligned} 16t^2\|B(u, v)\|^2 &= \|B(u + tv, u + tv)\|^2 + \|B(u - tv, u - tv)\|^2 \\ &\quad - 2(B(u + tv, u + tv), B(u - tv, u - tv)). \end{aligned}$$

Comparing coefficients of  $t^2$  gives the result.

**Theorem 2** *For any harmonic manifold  $(M, g)$  the sectional curvature  $K_M$  satisfies*

$$-2c^2 \leq K_M \leq c^2, \text{ where } c \text{ is a suitable constant.}$$

**Proof:** Embed the harmonic manifold  $(M, g)$  into  $L^2(M)$  such that all the geodesics are congruent screw lines in  $L^2(M)$ . Let  $B$  be the second fundamental form of  $M$ , then  $\|B(u, u)\| = \text{constant}(c, \text{ say})$ , for all unit vectors  $u$  at all points. Let  $u, v$  be orthonormal unit tangent vectors to  $M$  at a point. Let  $P$  be the plane generated by  $u, v$ . The sectional curvature is then,

$$K(P) = (B(u, u), B(v, v)) - \|B(u, v)\|^2$$

. using the above lemma one gets

$$K(P) = \frac{3}{2}(B(u, u), B(v, v)) - \frac{1}{2}c^2$$

Schwartz inequality gives  $-c^2 \leq (B(u, u), B(v, v)) \leq c^2$  which gives

$$-2c^2 \leq K(P) \leq c^2.$$

The inequality on the left holds iff  $B(u, u) = -B(v, v)$  and the one on the right holds iff  $B(u, u) = B(v, v)$ .

**Corollary 2.1** *Let  $M \subseteq S^n$  have all geodesics congruent screw lines in  $\mathbb{R}^{n+1}$ . Suppose  $\text{codim}M$  ( in  $\mathbb{R}^{n+1}$  ) = 1, 2 then  $M$  must be of constant curvature.*

**Proof:** Let  $\text{codim}M = 1$ . In this case  $\dim W = 1$  in the above lemma. Let  $\underline{e}$  be a unit vector in  $W$ , then we have  $B(u, v) = \pm (u, v) \underline{e}$ . Hence  $K(P) = c^2$  a constant. If  $\text{codim}M = 2$  we argue as follows. Let  $p$  be any point of  $M$ .  $p$  is normal to  $M$ , let  $\underline{n}$  be the other normal to  $M$ . Then for any tangent vector  $u$  to  $M$   $(B(u, u), p) = (\gamma''(0), \gamma(0))$  where  $\gamma(t)$  is a geodesic at  $p$ . But  $(\gamma''(0), \gamma(0))$  is a constant independent of the point  $p$  since all geodesics are congruent screw lines. Similarly  $(B(u, u), \underline{n})$  is also a constant independent of the point  $p$ . Again  $M$  has constant curvature.

## References

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